Math 335 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through §7.3

1. Assume $a_n \geq 0$ for all $n \geq 1$. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges. Give an example of a sequence $a_n \geq 0$ such that $\sum_1^\infty \sqrt{a_n a_{n+1}}$ converges and $\sum_1^\infty a_n$ diverges.

2. Prove that if $\sum_1^\infty a_n$ converges then $\sum_1^\infty \frac{\sqrt{a_n}}{n}$ converges. (Assume $a_n \geq 0$.)

3. Let x_n be a convergent sequence and let $c = \lim_{n\to\infty} x_n$. Let p be a fixed positive integer and let $a_n = x_n - x_{n+p}$. Prove that $\sum a_n$ converges and
$$\sum_1^\infty a_n = x_1 + x_2 + \ldots x_p - pc.$$

4. Suppose $a_n > 0$, $b_n > 0$ for all $n > 1$. Suppose that $\sum_1^\infty b_n$ converges and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Prove that $\sum_1^\infty a_n$ converges.

5. Let S be the set of all positive integers whose decimal representation does *not* contain 2. Prove that $\sum_{n\in S} \frac{1}{n}$ converges.

6. Suppose that $a_n \geq 0$ and $\sum_{n=0}^\infty a_n$ diverges; and suppose that $\sum_{n=0}^\infty a_n x^n$ converges for $|x| < 1$. Prove
$$\lim_{x\to 1^-} \sum_{n=0}^\infty a_n x^n = +\infty.$$

7. Suppose f_n is a sequence of continuous functions that converges uniformly on a set W. Let p_n be a sequence of points in W that converges to a point $p \in W$. Prove that $\lim_{n\to\infty} f_n(p_n) = f(p)$.

8. Let be a sequence of continuous functions in $I = [a, b]$ and suppose $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in I$. Suppose $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in I$ (point-wise convergence to 0). Is the convergence uniform? Give a proof or a counterexample.

9. Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{x}{(1+|x|)^n}$ converges for all x, but the convergence is not uniform.

10. Suppose $a_n > b_n > 0$, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Does $\displaystyle\sum_1^{\infty} (-1)^n b_n$ converge? Give a proof or a counterexample.

11. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges uniformly for $x \in [a, b], 0 < a < b < 2\pi$, but does not converge absolutely for any x.

12. Prove that $\displaystyle\sum_1^{\infty} (-1)^n \frac{\sin nx}{n}$ converges uniformly on $\{|x| < 1\}$ to a continuous function.

13. Let f_n be a sequence of functions defined on the open interval (a, b). Suppose $\lim_{x\to a^+} f_n(x) = a_n$ for all n. Suppose $\displaystyle\sum_1^{\infty} f_n$ converges uniformly on (a, b) to a function f. Prove that $\displaystyle\sum_1^{\infty} a_n$ converges and $\displaystyle\lim_{x\to a^+} f(x) = \sum_1^{\infty} a_n$. Do not assume f_n is continuous on (a, b).

14. Suppose the series $\sum_1^{\infty} a_n$ converges. Prove that $\sum_1^{\infty} \frac{a_n}{n^x}$ converges for $x \geq 0$. Let $f(x) = \sum_1^{\infty} \frac{a_n}{n^x}$. Prove that $\lim_{x\to 0^+} f(x) = \sum_1^{\infty} a_n$.

15. Let $p_j(t) = e^{-t}\dfrac{t^j}{j!}$.

    (a) Suppose $\sum_0^{\infty} a_n$ converges. Let $s_n = \sum_0^n a_j$. Prove that

    $$\lim_{t\to\infty} \sum_0^{\infty} s_j p_j(t) = \sum_0^{\infty} a_n.$$

    (b) Compute this limit in the case that $a_n = x^n$ for those x for which the limit exists (even in the case that $\sum x^n$ does not converge). This limit is called the Borel regularized value. What does this give for the *Borel regularized value* of $1 - 2 + 4 - 8 + 16 \pm \ldots$?

16. You will need to know the definitions of the following terms and statements of the following theorems.

(a) Convergence and divergence of a series
(b) Comparison test
(c) Integral test
(d) Cauchy condensation test
(e) Root test and ratio test
(f) Absolute and conditional convergence of a series
(g) Dirichlet's test
(h) Abel's theorem
(i) Uniform convergence of a sequence or series of functions
(j) Weierstrass M-test
(k) Continuity of a uniform limit of continuous functions
(l) Integration and differentiation of a sequence or series
(m) Power series
(n) Radius of convergence of a power series
(o) Integration and differentiation of a power series

17. There may be homework problems or example problems from the text on the midterm.